-trees and laminations for free groups III: Currents and dual -tree metrics

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1 Introduction

A geodesic lamination $\mathfrak{L}$ on a closed hyperbolic surface S, when provided with a transverse measure μ, gives rise to a "dual $\mathbb{R}$-tree" T_μ, together with an action of $G = \pi_1 S$ on T_μ by isometries. A point of T_μ corresponds precisely to a leaf of the lift $\widetilde{\mathfrak{L}}$ of $\mathfrak{L}$ to the universal covering $\widetilde{S}$ of S, or to a complementary component of $\widetilde{\mathfrak{L}}$ in $\widetilde{S}$. The G-action on T is induced by the G-action on $\widetilde{S}$ as deck transformations. This construction is well known (see [Mor86]). It is also known [Sko96] that conversely, for every small isometric action of a surface group $G = \pi_1 S$ on a minimal $\mathbb{R}$-tree T there exists a "dual" measured lamination $(\mathfrak{L}, \mu)$ on S, i.e. one has $T = T_\mu$ up to a G-equivariant isometry.

This beautiful correspondence has tempted geometers and group theorists to investigate possible generalizations, and the first one arises if one replaces the closed surface by a surface with boundary, and correspondingly the surface group G by a free group F_N of finite rank $N \geq 2$. A first glimpse of the potential problems can be obtained from two simultaneous but distinct identifications $F_N \xrightarrow{\cong} \pi_1 S_1$ and $F_N \xrightarrow{\cong} \pi_1 S_2$, thus obtaining actions of $\pi_1 S_1$ on a tree T_2 which are dual to a measured lamination on S_2, but in general not dual to any measured lamination on the surface S_1.

Worse, using the index of an $\mathbb{R}$-tree action by F_N as introduced in [GL95], it is easily seen that for many (perhaps even "most") small or very small $\mathbb{R}$-trees T with isometric F_N-action there is no identification whatsoever of F_N with the fundamental group of any surface that would make T dual to a lamination. An example of such trees are the forward limit trees T_α of certain irreducible automorphisms with irreducible powers (so called *iwip* automorphisms) of F_N. Much like pseudo-Anosov surface homeomorphisms, such an iwip automorphism has precisely one forward and one backward limit tree, T_α and $T_{\alpha^{-1}}$ respectively, and it induces a North-South dynamics on the space $\overline{CV}_N$ of projectivized very small F_N-actions on $\mathbb{R}$-trees (see [LL03]). Note that, contrary to the case of pseudo-Anosov homeomorphisms, it is a frequent occurence for an iwip automorphism α (see Corollary 5.7 below) that its stretching factor λ_α is different from the stretching factor $\lambda_{\alpha^{-1}}$ of its inverse.

In [CHL-II] for any $\mathbb{R}$-tree T with isometric F_N-action, a *dual lamination* $L(T)$ has been defined, which is the generalization of the geodesic lamination $\mathfrak{L}$ for a surface tree T_μ as discussed above. The goal of the present paper is to investigate the effect of putting an invariant measure μ on the dual lamination $L(T)$, or, in the proper technical terms, considering a free group current μ with support contained in $L(T)$. We prove here, if the F_N-action on T is very small and has dense orbits, that such a current defines indeed an induced measure on the metric completion $\overline{T}$ of T.

In the special case considered above where $T = T_\mu$ is dual to a measured lamination $(\mathfrak{L}, \mu)$ on a surface, then the transverse measure μ defines indeed a current on $L(T_\mu)$, and the induced measure on $\overline{T}_\mu$ defines a dual distance on T_μ which is precisely the same as the original distance on T_μ (i.e. the one that comes from the transverse measure μ on $\mathfrak{L}$). For arbitrary very small trees T with dense F_N-orbits, the measure on $\overline{T}$ induced by a current μ on $L(T)$ defines also a metric on T, except that this *dual metric* d_μ may in general be in various ways degenerate (compare §5 below). In particular, the dual distance may well be infinite for any two distinct points of T. Alternatively, it could be zero throughout the interior T of $\overline{T}$.

The main result of this paper is to show that these "exotic" phenomena are not just theoretically possible, but that they actually do occur in important classes of examples.

Let $\alpha \in Aut(F_N)$ be an iwip automorphism, let T_α be the forward limit tree of α. Then the dual lamination $L(T_\alpha)$ is uniquely ergodic (see Proposition 5.6): it carries a projectively unique non-trivial current μ. In this case the dual metric d_μ is simply called the *dual distance* d_* on T_α or on $\overline{T}_\alpha$.

Theorem 1.1. *Let $\alpha \in Aut(F_N)$ be an iwip automorphism with $\lambda_\alpha \neq \lambda_{\alpha^{-1}}$. Then the dual distance d_* on the forward limit tree T_α is zero or infinite throughout T_α.*

Geodesic currents have been introduced by F. Bonahon for hyperbolic manifolds [Bon86, Bon88]. They turn out to be a powerful tool, and they also admit generalizations to a much larger setting, compare [Fur02]. For free groups and their automorphisms, the first serious application was given in the thesis of M. Bestvina's student R Martin [Mar95]. Recently, I. Kapovich rediscovered currents and studied them systematically, see [Kap04, Kap03]. As Kapovich's papers are very carefully written and very accessible to non-experts, we will review geodesic currents here only briefly and refer for all of the basic detail work to the papers of Kapovich.

The novelty in the setup presented here is the relationship between currents and laminations, which we establish systematically through studying, for any current μ, the *support* $\mathrm{Supp}(\mu)$. The latter belongs to the space $\Lambda(F_N)$ of laminations for the free group F_N, which has been defined and investigated in detail in [CHL-I]. This gives a rather natural map from the space $\mathrm{Curr}(F_N)$ of currents to the space $\Lambda(F_N)$. The space $\mathrm{Curr}(F_N)$ of currents μ, as well as the resulting compact space $\mathbb{P}\mathrm{Curr}(F_N)$ of projectivized currents $[\mu]$, admit a natural action of the group $\mathrm{Out}(F_N)$ of outer automorphisms of F_N. The results derived in Proposition 3.1 and in Lemmas 3.2, 3.3 and 3.4 can be summarized as follows:

Theorem 1.2. *The map $Supp : Curr(F_N) \to \Lambda(F_N)$ induces a map*

$$\mathbb{P}Supp : \mathbb{P}Curr(F_N) \to \Lambda(F_N)$$

which has the following properties:

1. *$\mathbb{P}Supp$ is $Out(F_N)$-invariant.*
2. *$\mathbb{P}Supp$ is not injective.*
3. *$\mathbb{P}Supp$ is not surjective. However, every lamination $L \in \Lambda(F_N)$ possesses a sublamination $L_0 \subset L$ which belongs to the image of the map $\mathbb{P}Supp$.*
4. *$\mathbb{P}Supp$ is not continuous. However, if $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of currents which converges to a current μ, then the sequence of algebraic laminations $\mathbb{P}Supp([\mu_n])$ has at least one accumulation point in $\Lambda(F_N)$, and any such accumulation point is a sublamination of $\mathbb{P}Supp([\mu])$.*

Summing up, we believe that the results presented in this paper can be interpreted as follows:

On the one hand, the complete correspondence between small $\mathbb{R}$-trees and measured laminations, as known from the surface situation, does not fully extend to the world of free groups, very small $\mathbb{R}$-tree actions and currents. Unexpected degenerations seem to occure almost as a rule, and much further research is needed before one can speak of a "true understanding".

On the other hand, the spaces of currents, of $\mathbb{R}$-tree actions, and of algebraic laminations for F_N are naturally related, and although this relationship is more challenging than in the surface case, there is clearly enough interesting structure there to justify further research efforts. A small such further contribution has already been given, in [CHL05], where algebraic laminations where used to characterize $\mathbb{R}$-trees up to F_N-equivariant variations of their metric.

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2 Currents on F_N

Let $\mathcal{A}$ be a basis of the free group F_N or finite rank $N \geq 2$, and let $F(\mathcal{A})$ denote the set of finite reduced words in $\mathcal{A}^{\pm 1}$, which is as usually identified with F_N. A *geodesic current* for a free group F_N can be defined in various ways. In particular, there are the following three equivalent ways to introduce them:

I. Symbolic dynamist's choice: Consider the space $\Sigma_{\mathcal{A}}$ of biinfinite reduced indexed words $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ in $\mathcal{A}^{\pm 1}$, provided with the product topology, the shift operator $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, and with the involution $Z \mapsto Z^{-1}$, see [CHL-I]. A geodesic current is a non-trivial σ-invariant finite Borel measure μ on $\Sigma_{\mathcal{A}}$. We also require that μ is *symmetric*: the measure is preserved by the involution of $\Sigma_{\mathcal{A}}$ given by the inversion $Z \mapsto Z^{-1}$.

II. Geometric group theorist's choice: Consider the space $\partial^2 F_N$ of pairs (X, Y) of boundary points $X \neq Y \in \partial F_N$, endowed with the "product" topology, with the canonical diagonal action of F_N, and with the flip involution

$(X, Y) \mapsto (Y, X)$ as specified in [CHL-I]. A geodesic current is a non-trivial F_N- and flip-invariant Radon measure μ on $\partial^2 F_N$, i.e. a Borel measure that is finite on any compact set.

III. Combinatorist's choice: A geodesic current is given by a non-zero function $\mu : F_N = F(\mathcal{A}) \to \mathbb{R}_{\geq 0}$ with $\mu(w^{-1}) = \mu(w)$ for all $w \in F(\mathcal{A})$, which satisfies the left and the right *Kolmogorov property*: For all reduced words $w = y_1 \dots y_k \in F(\mathcal{A})$ one has

$$\mu(w) = \sum_{y \in \mathcal{A} \cup \mathcal{A}^{-1} \smallsetminus \{y_k^{-1}\}} \mu(wy) = \sum_{y \in \mathcal{A} \cup \mathcal{A}^{-1} \smallsetminus \{y_1^{-1}\}} \mu(yw).$$

This three viewpoints correspond to the three equivalent definitions given in [CHL-I] of a lamination for the free group F_N. We assume some familiarity of the reader with these three settings and will freely consider that a lamination is altogether symbolic (viewpoint I), algebraic (viewpoint II) and, a laminary language (viewpoint III). Whenever necessary, we emphasize the particular viewpoint used, by notationally specifying the lamination L in question as symbolic lamination $L_{\mathcal{A}}$, algebraic lamination L^2, or as laminary language $\mathcal{L}$ respectively.

For currents, the transition between the three viewpoints is also canonical (see [Kap04]), and we will freely move from one to the other without always notifying the reader. To be specific, the Kolmogorov value $\mu(w)$ of a reduced word $w = y_1 \dots y_k \in F(\mathcal{A})$, from the viewpoint III, is precisely the measure of the *cylinder*

$$C_{\mathcal{A}}(w) = \{\dots z_{i-1} z_i z_{i+1} \dots \mid z_1 = y_1, \dots, z_k = y_k\} \subset \Sigma_{\mathcal{A}}$$

from viewpoint I, and also, corresponding to viewpoint II, equal to the measure of the *algebraic cylinder* $C^2_{\mathcal{A}}(w) \subset \partial^2 F_N$ given by

$$\{(X, wX') \mid X = x_1 x_2 \dots, X' = x'_1 x'_2 \dots \in \partial F(\mathcal{A}), x_1 \neq y_1, x'_1 \neq y_k^{-1}\} \; .$$

Note that the algebraic cylinder $C^2_{\mathcal{A}}(w)$ is the image of the "symbolic" cylinder $C_{\mathcal{A}}(w)$ under the map $\Sigma_{\mathcal{A}} \to \partial^2 F_N$, $Z = Z_- \cdot Z_+ \mapsto (Z_-^{-1}, Z_+)$.

Remark 2.1. The reader should notice that in viewpoints I and III a basis $\mathcal{A}$ of F_N is crucially used, while II is "algebraic". It is very important to remember that basis change induces on the Kolmogorov function a more complicated operation than just rewriting the given group element in the new basis $\mathcal{B}$. The correct transition is given, for any reduced word $w \in F(\mathcal{B})$, by decomposing the algebraic cylinder $C^2_{\mathcal{B}}(w) \subset \partial^2 F_N$ into a finite disjoint union of translates $u_i C^2_{\mathcal{A}}(v_i)$ of properly chosen algebraic cylinders $C^2_{\mathcal{A}}(v_i)$, with $u_i, v_i \in F(\mathcal{A})$, and posing:

$$\mu(w) = \sum_i \mu(v_i).$$

Similarly as for laminations (see §1 of [CHL-I]), every element w of $F_N \smallsetminus \{1\}$ (or rather, every non-trivial conjugacy class) defines an *integer* current μ_w, given (in the language of viewpoint I) as follows: If $w = u^m$ for the maximal exponent $m \geq 1$, then the measure $\mu_w(C)$ of any measurable set $C \subset \Sigma_{\mathcal{A}}$ is equal to m times the number of elements of $C \cap L_{\mathcal{A}}(u)$, where $L_{\mathcal{A}}(u)$ is the finite set of biinfinite words of type $\ldots vv \cdot vv \ldots$, and $v \in F(\mathcal{A})$ is any of the cyclically reduced words conjugated to u or to u^{-1}. Alternatively, (in the language of viewpoint II) the current μ_w is given by an F_N-equivariant Dirac measure μ_w on $\partial^2 F_N$, defined as follows: For every measurable set $C^2 \subset \partial^2 F_N$ the value of $\mu_w(C^2)$ is given by the number of cosets $g < w > \subset F_N$ which contain an element v that satisfies $v(w^{-\infty}, w^{\infty}) \in C^2$ or $v(w^{\infty}, w^{-\infty}) \in C^2$. A third equivalent definition of μ_w (corresponding to viewpoint III) is given by a count of "frequencies", see [Kap03]. The noteworthy fact that μ_w depends only on the element $w \in F_N$ and not on the word $w \in F(\mathcal{A})$ is obvious in the second of these definitions, but rather puzzeling if one considers only the first or the third.

A current is *rational* if it is a non-negative linear combination of finitely many integer currents.

Remark 2.2. The above setup of the concept of currents in its various equivalent forms, together with the canonical identification $F_N = F(\mathcal{A})$ for any basis $\mathcal{A}$ of F_N, provides the ideal means to see very elegantly that many of the classical measure theoretic tools from symbolic dynamics do not depend on the underlying combinatorics of the chosen alphabet, but are rather algebraic in their true nature. Determining the exact point to which ergodic theory tools can be "algebraicized" seems to be a worthy task but goes beyond the scope of this paper.

3 The space Curr(F_N)

The set of currents on F_N will be denoted Curr(F_N). It comes naturally with several interesting structures, which we will discuss briefly in this section. We would like to stress that this space, as well as its projectivization, appears to be a very interesting and useful tool for many fundamental questions about automorphisms of free groups, and we expect that it will play an important role in the future developpement of this subject.

First, the set Curr(F_N) of currents carries the weak topology, which for any basis $\mathcal{A}$ of F_N is induced by the canonical embedding of Curr(F_N) into the vector space $\mathbb{R}^{F(\mathcal{A})}$, given by $\mu \mapsto (\mu(w))_{w \in F(\mathcal{A})}$. In particular, a family of currents μ_i converges towards a current $\mu \in \mathrm{Curr}(F_N)$ if and only if $\mu_i(w)$ converges to $\mu(w)$ for every $w \in F(\mathcal{A})$.

Next, the same formalism as explained in Remark 2.1 for a basis change defines canonically an action by homeomorphisms of Out(F_N) on the space Curr(F_N), which is formally given, for any $\alpha \in \mathrm{Aut}(F_N)$ and any $\mu \in \mathrm{Curr}(F_N)$, by $\alpha_*(\mu)(C) = \mu(\alpha^{-1}(C))$, for every measurable set $C \subset \partial^2 F_N$. This convention defines a left action of Out(F_N):

$$\alpha_*(\beta_*(\mu))(C) = \beta_*(\mu)(\alpha^{-1}(C)) = \mu(\beta^{-1}(\alpha^{-1}(C)))$$

$$= \mu((\alpha\beta)^{-1}(C)) = (\alpha\beta)_*(\mu)(C)$$

For any integer current μ_w, with $w \in F_N \smallsetminus \{1\}$, this gives (compare [Kap03, Kap04]):

$$\alpha_*(\mu_w) = \mu_{\alpha(w)}$$

Every current μ defines naturally a lamination $L(\mu)$ for the free group F_N. $L(\mu)$ can be viewed as an algebraic lamination $L^2(\mu)$, i.e. a non-empty subset of $\partial^2 F_N$ which is closed and invariant under the F_N-action and the flip involution, compare [CHL-I]. In this setting, $L^2(\mu) \subset \partial^2 F_N$ is simply the support Supp(μ) of the Borel measure μ on $\partial^2 F_N$, i.e. the complement of the biggest open set (= the union of all open sets) with measure 0. Alternatively, $L(\mu)$ is given via its laminary language $\mathcal{L}(\mu) = \{w \in F(\mathcal{A}) \mid \mu(w) > 0\}$. We refer to a current μ with support contained in a lamination L simply as *an invariant measure* on L. Alternatively, one says that μ is *carried by* L.

A lamination L which has, up to scalar multiples, only one current μ with support $L(\mu) = L$, is called *uniquely ergodic*. The simplest examples of non-uniquely ergodic laminations are given by the union L of two disjoint laminations L_0 and L_1, such that L_0 and L_1 are the support of currents μ_0 and μ_1 respectively (for example rational laminations $L_0 = L(a)$ and $L_1 = L(b)$ for distinct basis elements $a, b \in \mathcal{A}$). For $0 < \lambda < 1$ one obtains an interval of pairwise projectively distinct currents

$$\mu(\lambda) = \lambda\mu_1 + (1-\lambda)\mu_0\,,$$

all with support L.

Proposition 3.1. *Recall that we assume $N \geq 2$, and let $\Lambda(F_N)$ denote the space of laminations for F_N as introduced in [CHL-I]. The map*

$$Supp : Curr(F_N) \to \Lambda(F_N),\ \ \mu \mapsto L(\mu)$$

is $Out(F_N)$-equivariant, but not continuous and not surjective.

Proof. The $\mathrm{Out}(F_N)$-equivariance is a direct consequence of the definition of the action of $\mathrm{Out}(F_N)$ on $\mathrm{Curr}(F_N)$ and on $\Lambda(F_N)$.

To see that the map Supp is non-surjective it suffices to consider the symbolic lamination $L = L_{\{a,b\}}(Z)$ generated by the biinfinite word $Z = \ldots aaab \cdot aaa \ldots$. It consists of the σ-orbit of Z and of the periodic word $\ldots aa \cdot aa \ldots$, as well as of their inverses. However, it is an easy exercise to show that any Kolmogorov function μ on the associated laminary language $\mathcal{L}_{\{a,b\}}(Z)$, as it takes on values in $\mathbb{R}_{\geq 0}$ and not in $\mathbb{R}_{\geq 0} \cup \{\infty\}$, must associate the value 0 to any word that contains the letter b, so that all the measure of μ will be concentrated on the sublamination $L_{\{a,b\}}(a)$ of L.

The fact that the map Supp is non-continuous can be seen from the above defined family $\mu(\lambda)$ of currents with constant support L, by letting the parameter λ converge inside the open interval $(0, 1)$ to the value 0 (or 1): For any such λ the support of $\mu(\lambda)$ is clearly the union $L_0 \cup L_1$, while for the limit one gets $L(\mu(0)) = L_0$ (or $L(\mu(1)) = L_1$). $\square$

The space $\mathrm{Curr}(F_N)$ has some additional structures which are not matched by corresponding structures in $\Lambda(F_N)$. For example, there is a canonical linear structure on $\mathrm{Curr}(F_N)$, given simply by the embedding of $\mathrm{Curr}(F_N)$ into the real vector space $\mathbb{R}^{F(\mathcal{A})}$. Projectivization $\mu \mapsto [\mu]$ defines the space of *projectivized currents* $\mathbb{P}\mathrm{Curr}(F_N)$. Both $\mathrm{Curr}(F_N)$ and its projectivization are infinite dimensional, but $\mathbb{P}\mathrm{Curr}(F_N)$ is compact. Clearly, the map Supp splits over the projectivization, thus inducing a map $\mathbb{P}\mathrm{Supp} : \mathbb{P}\mathrm{Curr}(F_N) \to \Lambda(F_N)$, which by Proposition 3.1 is $\mathrm{Out}(F_N)$-equivariant, non-continuous, and non-surjective. We obtain furthermore

Lemma 3.2. *The map $\mathbb{P}Supp : \mathbb{P}Curr(F_N) \to \Lambda(F_N)$ is non-injective.*

Proof. Any non-uniquely ergodic lamination, in particular the above defined family $\mu(\lambda)$ of currents with constant support L, shows that the map $\mathbb{P}$Supp is not injective. $\square$

A second interesting example for the non-continuity of the map Supp, other than the one given in the proof of Proposition 3.1, is given by the rational currents $\frac{1}{n}\mu_{ab^n}$ which converge to μ_b, while their support $L(ab^n)$ converge to the lamination generated by $\dots bba \cdot bb \dots$ and $\dots bb \cdot bb \dots$, which is strictly larger than the lamination $L(b)$.

This last example, as also the one given in the proof of Proposition 3.1, indicates that a weaker statement than the continuity might be true for the map Supp. Since this will be needed in §5 as an important ingredient for the proof of Proposition 5.6, we formalize it here:

We say that a subset δ of $\Lambda(F_N)$ is *saturated* if δ contains with any lamination also all of its sublaminations.

Lemma 3.3. *Let $\delta \subset \Lambda(F_N)$ be a closed saturated subset of laminations. Then the full preimage $\Delta \subset \mathbb{P}Curr(F_N)$ of δ under the map $\mathbb{P}Supp$ is closed.*

Proof. We consider a sequence of currents μ_k in Curr(F_N), with $L(\mu_k) \in \delta$ for any μ_k. By the compactness of $\mathbb{P}$Curr(F_N) and of $\Lambda(F_N)$ we can assume, after possibly passing over to a subsequence, that there is a current $\mu \in$ Curr(F_N) and a lamination $L \in \Lambda(F_N)$ with $[\mu] = \lim\limits_{k\to\infty} [\mu_k]$ and $L = \lim\limits_{k\to\infty} L(\mu_k)$. By properly normalizing the μ_k we can actually assume that $\mu = \lim\limits_{k\to\infty} \mu_k$.

We now fix a basis $\mathcal{A}$ of F_N and consider the value of the Kolmogorov function $\mu(w)$ for any $w \in F_N \smallsetminus \{1\}$. If $\mu(w) > 0$, then by the topology on Curr(F_N), for any ε with $\mu(w) > \varepsilon > 0$ there is a bound k_0 such that for any $k \geq k_0$ one has $|\, \mu_k(w) - \mu(w)\, | < \varepsilon$. This shows for all $k \geq k_0$ that w belongs to the laminary language $\mathcal{L}(\mu_k)$. But this implies that w belongs to the laminary language of L, which shows that μ is carried by L. Since by hypothesis δ is closed and saturated, this shows that $[\mu]$ is contained in Δ, so that the latter must be closed. $\square$

A weaker statement than the surjectivity of the map Supp is crucially used in §5, again in the proof of Proposition 5.6:

Lemma 3.4. *Every lamination $L \in \Lambda(F_N)$ contains a sublamination which is the support of some current $\mu \in Curr(F_N)$.*

Proof. For some basis $\mathcal{A}$ of F_N, let $Z = \dots z_{i-1} z_i z_{i+1} \dots$ be a leaf of the lamination L. Let $Z_n = z_{-n} \dots z_n$ be the central subword of Z of length $2n+1$.

For every $n \in \mathbb{N}$ we define a "counting function" $m_n : F(\mathcal{A}) \to \mathbb{R}_{\geq 0}$, by setting, for any word w in $F(\mathcal{A})$, $m_n(w)$ to be the number of occurences of w as subword of Z_n or of Z_n^{-1}, divided by $4n+2$. It follows directly that m_n satisfies the equations that defines the right and the left Kolmogorov property, up to possibly an error of absolute value less than $\frac{1}{2n+1}$. The total value of m_n on the set of words of length 1 is 1, for any $n \in \mathbb{N}$. Moreover $m_n(w)$ is non-zero only for subwords of Z.

For each word w in $F(\mathcal{A})$ we can chose a subsequence of $(m_n)_{n \in \mathbb{N}}$ whose value at w converges. By a diagonal argument we get a subsequence that converges pointwise to a limit function μ which satisfies the Kolmogorov laws while still having total value 1 on set of words of length 1, so that it is non-zero.

By construction, we have $m_n(w) = m_n(w^{-1})$ for all $w \in F(\mathcal{A})$, so that the same is true for μ. Hence μ is a current. Its support is contained in the set of subwords of Z and thus, as a lamination, in L. □

A very interesting subspace $\mathcal{M} \subset \mathbb{P}\mathrm{Curr}(F_N)$ has been introduced by R. Martin in [Mar95] as closure of the $\mathrm{Out}(F_N)$-orbit of $[\mu_a]$, for any element a of any basis $\mathcal{A}$ of F_N. R. Martin shows that a projectivized integer current $[\mu_w]$ belongs to $\mathcal{M}$ if and only if w is contained in a proper free factor of F_N. In contrast to the analogous situation for $\overline{\mathrm{Out}(F_N) L(a)}$ (compare Proposition 8.1 of [CHL-I]), for $N \geq 3$ it has been shown in [KL06], Theorem B, that $\mathcal{M}$ is the unique minimal subspace of $\mathrm{Curr}(F_N)$ which is non-empty, closed and $\mathrm{Out}(F_N)$-invariant.

The fact that currents behave somehow more friendly than laminations is underlined by the following fact, proved in R. Martin's thesis and attributed there to M. Bestvina (compare to Proposition 6.5 of [CHL-I]):

Proposition 3.5 ([Mar95]). *The set of projectivized integer currents $[\mu_w]$, for any $w \in F_N$, is dense in $\mathbb{P}Curr(F_N)$.*

4 Geometric currents

A large class of very natural examples for a current $\mu \in \mathrm{Curr}(F_N)$ is given by any geodesic lamination $\mathfrak{L} \subset S$, provided with a transverse measure μ', where S is a hyperbolic surface with boundary as considered in the section 3 of [CHL-I] and section 6 of [CHL-II]. In this case the measure μ on $\partial^2 F_N$ can be nicely seen geometrically through the canonical identification of ∂F_N with the space $\partial\widetilde{S}$ of ends of the universal covering $\widetilde{S}$, which is embedded as subset in the boundary at infinity $S^1_\infty = \partial\mathbb{H}^2$. Two disjoint intervals $A, B \subset S^1_\infty$, with intersections $A' = A \cap \partial\widetilde{S}, B' = B \cap \partial\widetilde{S}$, define a measurable set $A' \times B'$ of $\partial^2 F_N$, and the measure $\mu(A' \times B')$ is precisely given by the measure $\mu'(\beta)$ of an arc β in S which is transverse to $\mathfrak{L}$, and which lifts to an arc $\widetilde{\beta}$ in $\widetilde{S} \subset \mathbb{H}^2$ that has its two endpoints on the two extremal leaves of $\widetilde{\mathfrak{L}} \subset \widetilde{S}$ which bound the set of all leaves of $\widetilde{\mathfrak{L}}$ that have one endpoint in A and one endpoint in B.

5 The dual metric for $\mathbb{R}$-trees

In this section we assume familiarity of the reader with the notions of [CHL-II], from which we also import the notation without further explanations.

In the last section we have seen that every transverse measure μ on a geodesic lamination $\mathfrak{L}$ which is contained in a hyperbolic surface S, with non-empty boundary and with an identification $\pi_1 S = F_N$, gives rise to a canonical current in $\mathrm{Curr}(F_N)$ which we also denote by μ. In section 6 of [CHL-II] we have discussed that $(\mathfrak{L}, \mu)$ determines an $\mathbb{R}$-tree T_μ with isometric F_N-action, and that the support of the current μ and the dual lamination of T_μ are the same: this lamination is precisely the lamination associated to $\mathfrak{L} \subset S$.

One of the most intriguing aspects of the relationship between currents and $\mathbb{R}$-trees comes from the attempt to extend this correspondence, which for surfaces is almost tautological, to more general $\mathbb{R}$-trees T. Indeed, the goal of this section is to understand better the true nature of the interaction between the metric on T and an invariant measure μ carried by the dual lamination $L(T)$ as defined in [CHL-II].

In the sequel we consider the dual lamination $L(T)$ as algebraic lamination $L^2(T)$, i.e. a non-empty, F_N-invariant, flip-invariant and closed subset of

$\partial^2 F_N$. From [CHL-II] we know that there is a map $\mathcal{Q}^2 : L^2(T) \to \overline{T}$ which is F_N-equivariant and continuous (see Proposition 8.3 of [CHL-II]). Here T is an element of the boundary $\partial\mathrm{cv}_N$ of the unprojectivized Outer space $\mathrm{cv}_N :$ in particular, T is a non-trivial $\mathbb{R}$-tree with minimal, very small F_N-action by isometries (see [CHL-II], §2). We also require that the F_N-orbits of points are dense in T ("T has dense orbits"), and we denote by $\overline{T}$ the metric completion of T.

Corollary 5.1. *For all $T \in \partial cv_N$ with dense orbits, the map $\mathcal{Q}^2 : L^2(T) \to \overline{T}$ is measurable (with respect to the two Borel σ-algebras on $L^2(T)$ and on $\overline{T}$).* □

We apply the last corollary in order to define an *extended pseudo-metric* d_μ on $\overline{T}$, for any current μ which is carried by $L(T)$. An extended pseudo-metric is just like a metric, except that distinct points P, Q may have distance 0, positive distance, or distance ∞.

Definition 5.2. Let $T \in \partial cv_N$ be with dense orbits, and assume that $\mu \in \mathrm{Curr}(F_N)$ satisfies $\mathrm{Supp}(\mu) \subset L(T)$. One then defines, for any $P, Q \in \overline{T}$, their *μ-distance* as follows:

$$d_\mu(P, Q) = \mu((\mathcal{Q}^2)^{-1}([P, Q])) \ [= \mathcal{Q}^2_*(\mu)([P, Q])\,]$$

Clearly the function d_μ is symmetric and, since $\overline{T}$ is a tree, it satisfies the triangular inequality. For three points $P, Q, R \in \overline{T}$ with $Q \in [P, R]$ one has $d_\mu(P, R) = d_\mu(P, Q) + d_\mu(Q, R)$ unless $\mu((\mathcal{Q}^2)^{-1}(\{Q\})) > 0$, which of course can happen (for example if Q has non-trivial stabilizer which carries all of the support of μ).

We distinguish now three special cases (note that we always assume that T is a minimal $\mathbb{R}$-tree, so that it agrees with its interior): The metric d_μ is called *zero throughout* T if any two points in T have μ-distance 0. It is called *infinite throughout* T if any two distinct points in T have μ-distance ∞. It is called *positive throughout* T if any two distinct points in T have positive finite μ-distance. Otherwise we call the μ-distance *mixed*.

A particular case, which is of special importance, is the following:

Definition 5.3. An $\mathbb{R}$-tree $T \in \partial cv_N$ is called *dually uniquely ergodic* if the dual lamination $L(T)$ is uniquely ergodic.

We note that, in the case where T is dually uniquely ergodic, the μ-distance is uniquely determined by T, up to rescaling. In this case we suppress the measure μ and speak simply of the *dual distance* d_* on T.

Conjecture 5.4. *If T is dually uniquely ergodic then the dual distance is not mixed.*

We finish this article by proving that the case of dual distances which are infinite or zero throughout the interior does actually exist, and that it occurs in a natural context. We assume from now on a certain familiarity with some of the modern tools for the geometric theory of automorphisms of free groups. Background material and references can be found in [Vog02]. In particular we will use below the following facts and definitions:

Remark 5.5. (1) An automorphism α of F_N is called *irreducible with irreducible powers (iwip)* if no non-trivial proper free factor of F_N is mapped by any positive power of α to a conjugate of itself.
(2) It is known (compare [LL03]) that for every iwip automorphism α there is, up to F_N-equivariant homothety, precisely one minimal *forward limit* $\mathbb{R}$-tree $T_\alpha \in \partial cv_N$ which admits a homothety $H : T_\alpha \to T_\alpha$ with stretching factor $\lambda_\alpha > 1$ that twistedly commutes with α. By this we mean that

$$\alpha(w)H = Hw : T_\alpha \to T_\alpha$$

holds for every $w \in F_N$. Note that both, the map H as well as the F_N-action on T, extend canonically to the metric completion $\overline{T}_\alpha$, so that the last statement holds also for $\overline{T}_\alpha$ instead of T_α.
(3) In terms of the induced action of $Out(F_N)$ on the non-projectivized closed Outer space $\overline{cv}_N$ (see [CHL-II], §9), the equation in (2) can be expressed by stating

$$T\alpha_* = \alpha_*^{-1}T = \lambda_\alpha T\,,$$

where $\lambda_\alpha T$ denotes the tree T rescaled by the factor λ_α.
(4) As a consequence of the equation in (2), the homothety H satisfies:

$$H\mathcal{Q}^2 = \mathcal{Q}^2\alpha : L^2(T_\alpha) \to \overline{T}_\alpha\,.$$

(5) There is no further fixed point of the α_*-action on $\overline{CV}_N$ other than the points $[T_\alpha]$ and $[T_{\alpha^{-1}}]$ specified above. In [LL03] it is shown that any iwip automorphism has North-South dynamics on $\overline{CV}_N$.

(6) One knows from [Mar95], Theorem 30 (again attributed to M. Bestvina) that, if α is not *geometric*, i.e. induced by a surface homeomorphism $h : S \overset{\approx}{\to} S$ via some identification $F_N \cong \pi_1 S$, then the α_*-action on $\mathbb{P}\mathrm{Curr}(F_N)$ possesses precisely two fixed points, an attractive and a repelling one, and that α_* has a North-South dynamics on $\mathbb{P}\mathrm{Curr}(F_N)$.

(7) Let us denote by $\mu_\alpha \in \mathrm{Curr}(F_N)$ a representative of the attracting fixed point of the α_*-action on $\mathbb{P}\mathrm{Curr}(F_N)$. It satisfies $\alpha_*(\mu_\alpha) = \lambda_\alpha \mu_\alpha$, see [Mar95], where λ_α is the stretching factor given in (2).

(8) Following [Mar95], the support of μ_α is contained in the so called *legal lamination* $L_\alpha \in \Lambda(F_N)$: Its leaves are represented, for any train track representative $f : \tau \to \tau$ of α, by biinfinite legal paths in τ, and consequently by non-trivial (in fact: biinfinite) geodesics in T_α (compare with the *attractive lamination* defined in [BFH97]). In particular, it follows from the alternative definition of the dual lamination, $L(T) = L_{\mathcal{Q}}(T)$, given in Theorem 1.1 of [CHL-II], that the two laminations L_α and $L(T_\alpha)$ are disjoint.

(9) Any iwip automorphism possesses a train track representative $f : \tau \to \tau$ with transition matrix that is primitive. As a consequence, any edge e of τ will have an iterate $f^k(e)$ which crosses over all other edges. The canonical image in T_α (under the map $i : \widetilde{\tau} \to T_\alpha$, see [LL03]) of any lift of $f^k(e)$ to the universal covering $\widetilde{\tau}$ is a segment which has the property that the union of its F_N-translates covers all of T_α.

Proposition 5.6. *For every non-geometric iwip automorphisms $\alpha \in Aut(F_N)$, the forward limit tree T_α is dually uniquely ergodic.*

Proof. From the $\mathrm{Out}(F_N)$-equivariance of the map $\lambda^2 : \partial cv_N \to \Lambda(F_N)$ in Proposition 9.1 of [CHL-II], together with Remark 5.5 (3) above, it follows that the dual lamination $L(T_\alpha)$ is fixed by α. Hence the set $\Delta(\alpha) \subset \mathbb{P}\mathrm{Curr}(F_N)$, which consists of all preimages under the map $\mathbb{P}\mathrm{Supp}$ of the lamination $L(T_\alpha)$ and any of its sublaminations, is invariant under the action of α_* (by the equivariance of the maps Supp and $\mathbb{P}\mathrm{Supp}$, see Proposition 3.1). As the set of all sublaminations of a given lamination is closed, see Proposition 6.4 of [CHL-I], it follows from Lemma 3.3 that $\Delta(\alpha)$ is closed. Furthermore $\Delta(\alpha)$ is non-empty, by Lemma 3.4. Thus $\Delta(\alpha)$ is the non-empty union of closures of α_*-orbits, so that it must contain the closure of at least one α_*-orbit in $\mathbb{P}\mathrm{Curr}(F_N)$. From the North-South dynamics of the α_*-action on $\mathbb{P}\mathrm{Curr}(F_N)$ (Remark 5.5 (6)) it follows that either $\Delta(\alpha)$ consists of precisely one of the two fixed points $[\mu_\alpha]$ or $[\mu_{\alpha^{-1}}]$, or else it contains both of them.

But according to Remark 5.5 (8) the support of μ_α is contained in the legal lamination L_α, which in turn is disjoint from $L(T_\alpha)$. Hence $[\mu_\alpha]$ is not contained in $\Delta(\alpha)$, which proves that the latter consists precisely of the point $[\mu_{\alpha^{-1}}]$. This shows that $L(T_\alpha)$ supports only one (projectivized) current, namely $[\mu_{\alpha^{-1}}]$. $\square$

We can now give the proof of our main result as stated in §1:

Proof of Theorem 1.1. From Proposition 5.6 and its proof we know that the forward limit tree T_α has dual lamination $L(T_\alpha)$ which carries an (up to homothety) unique current, and that this current is equal to $\mu_{\alpha^{-1}}$.

We now calculate, for any $P, Q \in T_\alpha$ (using Remark 5.5 (4) to get the third, and (7) to get the sixth of the equalities below):

$$d(H(P), H(Q)) = \lambda_\alpha d(P, Q)$$

and

$$\begin{aligned} d_*(H(P), H(Q)) &= \mu_{\alpha^{-1}}((\mathcal{Q}^2)^{-1}([H(P), H(Q)])) \\ &= \mu_{\alpha^{-1}}((H\mathcal{Q}^2\alpha^{-1})^{-1}([H(P), H(Q)])) \\ &= \mu_{\alpha^{-1}}(\alpha((\mathcal{Q}^2)^{-1}([P, Q]))) \\ &= \alpha_*^{-1}(\mu_{\alpha^{-1}})((\mathcal{Q}^2)^{-1}([P, Q])) \\ &= \lambda_{\alpha^{-1}}\mu_{\alpha^{-1}}((\mathcal{Q}^2)^{-1}([P, Q])) \\ &= \lambda_{\alpha^{-1}} d_*(P, Q) \end{aligned}$$

Assume now that some points $P \neq Q \in T_\alpha$ have finite dual distance. By iterating H one finds an interval $[H^n(P), H^n(Q)]$ with the property that the union of its F_N-translates covers all of T_α (compare Remark 5.5 (9)). This implies that any two points in T_α have finite dual distance. If the dual distance function is furthermore non-zero, by the same argument it follows that any two points have non-zero distance. Thus the dual metric d_* on T_α defines a non-trivial $\mathbb{R}$-tree T_α^* with free F_N-action, and hence, since the equation in Remark 5.5 (2) carries over from T_α to T_α^*, the $\mathbb{R}$-tree T_α^* defines a fixed point $[T_\alpha^*]$ of the α_*-action on ∂CV_N (see §2 and §9 of [CHL-II]). By Remark 5.5 (5) the point $[T_\alpha^*]$ must agree with either $[T_\alpha]$ or $[T_{\alpha^{-1}}]$. But this cannot be because we computed above that the streching factor of the α_*-action on T_α^* is equal to $\lambda_{\alpha^{-1}}$ and hence bigger than 1 (which rules out $[T_\alpha^*] = [T_{\alpha^{-1}}]$), but different from λ_α (thus ruling out $[T_\alpha^*] = [T_\alpha]$, by Remark 5.5 (3)).

Hence the dual metric d_* must be either zero or infinite throughout T_α. $\square$

A concrete example of an automorphism that satisfies the properties stated in Theorem 1.1 as hypotheses is given in [ABHS05] by the automorphism

$$\begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

of F_3, which has streching factor $1,84\ldots$, while its inverse

$$\begin{array}{rcl} a & \mapsto & c \\ b & \mapsto & c^{-1}a \\ c & \mapsto & c^{-1}b \end{array}$$

has streching factor $1,39\ldots$.

An iwip automorphism $\alpha \in \mathrm{Aut}(F_N)$ is called *parageometric*, if α is not geometric, but T_α is a geometric tree (see [GL95, GJLL98]). It has been proved recently in [HM04], see also [Gui04], that in this case the iwip automorphism α^{-1} is not parageometric, and that its stretching factor $\lambda_{\alpha^{-1}}$ is strictly smaller than λ_α (compare [Gau05]). A family of such automorphisms, one for any $N \geq 3$, has been exhibited and investigated in[JL98]. We summarize:

Corollary 5.7. *The dual metric on the forward limit tree of any parageometric iwip automorphism of F_N, or of its inverse, is always infinite or zero throughout.*

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